# 1.1 Complex Numbers

In the past, you have not been able to solve some quadratic equations like $x^2 + 4 = 0$ and $x^2 + 1 = 0$, because there are no real numbers you can square to get a negative answer. To solve this issue, mathematicians created a new, expanded number system based on one new number. However, this was not the first time mathematicians had invented new numbers! To read about other such inventions, refer to the Historical Note that follows problem 1.

In this lesson, you will learn about imaginary numbers and how you can use them to solve equations you were previously unable to solve.

- **1.** Consider the equation $x^2 = 2$.

  a. How do you "undo" squaring a number?

  b. When you solve $x^2 = 2$, how many solutions should you get?

  c. How many $x$-intercepts does the graph of $y = x^2 - 2$ have?

  d. Solve the equation $x^2 = 2$. Write your solutions both as radicals and as decimal approximations.

**Historical Note: Irrational Numbers**

In Ancient Greece, people believed that all numbers could be written as fractions of whole numbers (what are now called **rational numbers**). Many individuals realized later that some numbers could not be written as fractions (such as $\sqrt{2}$ ), and these individuals challenged the accepted beliefs. Some of the people who challenged the beliefs were exiled or outright killed over these challenges!

The Greeks knew that for a one-unit square, the length of the diagonal, squared, yielded 2. When it was shown that no rational number could do that, the existence of what are called **irrational numbers** was accepted and symbols like $\sqrt{2}$ were invented to represent them.

The problem $x^2 = 3$ also has no rational solutions; fractions can never work exactly. The rational (i.e., decimal) solutions that calculators and computers provide are only approximations; the exact answer can only be represented in radical form, namely, $\pm\sqrt{3}$ .

- **2.** Mathematicians throughout history have resisted the idea that some equations may not be solvable. Still, it makes sense that $x^2 + 1 = 0$ cannot be solved because the graph of $y = x^2 + 1$ has no $x$-intercepts (and x-intercepts are the roots or solutions of an equation). What happens when you try to solve $x^2 + 1 = 0$?

## Historical Note: Imaginary Numbers

In some ways, each person's math education parallels the history of mathematical discovery. When you were much younger, if you were asked, *"How many times does 3 go into 8*?" or *"What is 8 divided by 3*?" you might have said, *"3 doesn't go into 8."* Then you learned about numbers other than whole numbers, and the question had an answer. Later, if you were asked, *"What number squared makes 5*?" you might have said, *"No number squared makes 5*." Then you learned about numbers other than rational numbers, and you could answer that question. Similarly, until about 500 years ago, the answer to the question, *"What number squared makes −1?"* was, *"No number squared makes −1."* Then something remarkable happened. An Italian mathematician named Bombelli used a formula for finding the roots of third-degree polynomials. Within the formula was a square root, and when he applied the formula to a particular equation, the number under the square root came out negative. Instead of giving up, he had a brilliant idea. He had already figured out that the equation had a solution, so he decided to see what would happen if he pretended that there *was* a number he could square to make a negative. Remarkably, he was able to continue the calculation, and eventually the "imaginary" number disappeared from the solution. More importantly, the resulting answer worked; it solved his original equation. This led to the acceptance of these so-called **imaginary numbers**. The name stuck, and mathematicians became convinced that all quadratic equations do have solutions. Of course, in some situations you will only be interested in real number solutions (that is, solutions not having an imaginary part).

- **3.** In the 1500s, an Italian mathematician named Rafael Bombelli invented the imaginary number $\sqrt{-1}$ , which is now called $i$. $\sqrt{-1} = i$ implies that $i^2 = -1$. After this invention, it became possible to find solutions for $x^2 + 1 = 0$; they are $i$ and $-i$.

  The value of $\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16i^2} = 4i$. Use the definition of $i$ to rewrite each of the following expressions.

  a. $\sqrt{-4}$ b. $(2i)(3i)$ c. $(2i)^2(-5i)$ d. $\sqrt{-25}$

### Part I: Introducing Imaginary Numbers

1. Solve the following equations:

   a. $x^2 = 121$ b. $5x^2 = 200$

2. How would we solve an equation like this? $x^2 = -36$

3. The imaginary number $i$ is introduced when we are asked to take the square root of a negative number.

   a. If we let $i = \sqrt{-1}$ , what is $i^2$? Show your work and explain each step.

4. Solve the following equations using your conclusions from above:

a. $x^2 = -100$ c. $x^2 = -49$ e. $x^2 = -18$ g. $2x^2 = -50$ i. $-3x^2 - 27 = 162$

b. $x^2 = -25$ d. $x^2 = -169$ f. $x^2 = -96$ h. $4x^2 - 4 = -68$ j. $5(x^2 + 20) = -300$

## Part II: Powers of $i$

Using the information about the imaginary unit $i$ you have learned above, calculate the following powers of $i$. The first two problems, $i^1$ *and* $i^2$, have been completed for you.

$i^1 = i$

$i^2 = -1$

$i^3 =$ _______

$i^4 =$ _______

$i^5 =$ _______

$i^6 =$ _______

$i^7 =$ _______

$i^8 =$ _______

⋮

$i^{100} =$ ______

$i^{101} =$ ______

Describe what patterns you recognize from completing the problems above.

## Part III: Complex Numbers and Operations

We can use addition and subtraction with complex numbers by combining like terms. Complete each problem below by writing an equivalent expression for each.

1. $(2 + 4i) + (5 - 7i)$
2. $(4 - 8i) - (3 - 6i)$
3. $3(-2 + 5i) - (1 - 7i)$
4. $(7 + \sqrt{-81}) + 17i$

We can also use multiplication with complex numbers. ***Remember:*** $\boldsymbol{i^2 = -1}$.
Complete each problem by writing an equivalent expression for each.

1. $(-6i)(-6i)$
2. $(-5i)(3i)$
3. $\sqrt{-6} \cdot \sqrt{-15} \cdot \sqrt{-80}$
4. $-9i(4 - 3i)$
5. $(4 - 6i)(7 + i)$
6. $(4 - 6i)(6 - 6i)$
7. $(-2i + 7)(-2i - 7)$
8. $(5 + 3i)(5 - 3i)$

## Summary

Imaginary numbers are part of a broader set of numbers called *complex numbers*. Using the word bank, fill in the diagram below, placing each type of number in its appropriate place. Give two examples of each type of number in each category.

***Word Bank***

Real Numbers
Imaginary Numbers
Complex Numbers
Rational
Irrational